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Extended phase spaces and twistor theory

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Abstract. A commuting Minkowski position variable in the two-twistor phase space is found, providing a link between twistor phase spaces and the extended phase spaces for an elementary spinning particle, as defined by Zakrzewski. The two-twistor phase space is shown to be the product of three symplectic manifolds: the (forward) cotangent bundle to the Minkowski spacetime, the cotangent bundle to a circle (electric charge phase space) and the cotangent bundle to the real projective spinor space. The decomposition of the latter into Lorentz-'irreducible' parts gives exactly the one-parameter family of extended phase spaces described by Zakrzewski for $b = 0$ (and arbitrary a).

0. Introduction

Well known troubles to formulate a relativistic mechanics of interacting particles may suggest that a non-standard approach to the subject is needed. As a possible way out, it has been proposed in [2] to consider arbitrary Cartesian powers of the twistor phase space (the latter being understood as a fundamental building block), which carry a natural action of the Poincaré group, and then to try to recognize the spacetime nature of such a (possibly extended) object (in addition to some purely internal degrees of freedom). The dynamics is governed by some Poincaré-invariant Hamiltonian.

It has been pointed out in [3, 2] that in the product of two or more twistor phase spaces, there exist natural spacetime coordinates X^a ($a = 0, 1, 2, 3$). According to [2], they do not commute, namely

$$\{X^a, X^b\} = -\frac{1}{m^2} R^{ab}$$

where m is the total mass and R is the *spin tensor*: the rotational (with respect to the linear momentum p) part of the Lorentz momentum (relative to any point). The position X described by X^a is interpreted as the *centre-of-mass* of the system, since it turns out that the Lorentz momentum M_X with respect to X is purely *internal*: it coincides with R (in other words, $M_X p = 0$).

On the other hand, in [1] a family of phase spaces extending the Souriau's [4] spaces of motions of elementary systems was introduced, in order to include explicitly the (commuting) spacetime position. We have called them *extended phase spaces*. Studying the relationship between the two-twistor phase space and extended phase spaces was the main motivation for the present paper.

§ Humboldt fellow.

The result of the study is as follows. We have found commuting position variables in the two-twistor phase space (see section 3 below). This allows us to represent the whole two-twistor phase space as a (symplectic) product of the cotangent bundle to the Minkowski space and some other symplectic manifold, described by those variables which commute with the total linear momentum and the (new) position. By the results of [1], a part of these variables is immediately given by the Lorentz momentum relative to the commuting position (in [1] it was denoted by S; here we denote it by Σ in order not to suggest its interpretation as the *spin*). We find that this second symplectic manifold is itself a product of a cotangent bundle to $U(1)$ (which may be interpreted as the Kaluza–Klein variable) and the cotangent bundle to the real projective spinor space.

The latter cotangent bundle decomposes (in the sense of symplectic reductions) onto Lorentz co-adjoint orbits with $b = 0$ and arbitrary a (in the notation of [1]). This explains our main result: apart from the Kaluza–Klein phase and the conjugate charge, the two-twistor phase space decomposes onto the one-parameter family of extended phase spaces, the parameter a of [1] being identified with the difference of helicities of the two twistors. It is this parameter which is identified with the 'charge' corresponding to the $U(1)$ action on the real projective spinor space (the Hopf $U(1)$-principal bundle over the 2-sphere). This explains the quantization of a (in the corresponding quantum theory).

1. Twistor space

The fundamental object in twistor theory [5] is the *twistor space* T, which is a four-dimensional complex vector space equipped with a Hermitian form

$$T \times T \ni (Z_1, Z_2) \mapsto (Z_1|Z_2) \in \mathbb{C}$$

(say, anti-linear in the second argument) of signature $(2, 2)$. Linear transformations of T preserving the Hermitian form and having the determinant equal 1 form a group G isomorphic to $SU(2, 2)$. The set of two-dimensional isotropic (with respect to the Hermitian form) subspaces in T is interpreted as the *compactified Minkowski space*. The ordinary Minkowski space M is identified with the set of two-dimensional isotropic subspaces which are transversal to one distinguished such subspace (the *infinity*) $S \subset T$.

Relaxing the isotropy condition, we obtain the complexified Minkowski space. Two-dimensional subspaces transversal to S are in one-to-one correspondence with projections on S and to each such projection there corresponds the conjugate—with respect to the Hermitian form—projection. This gives the involution in the complexified Minkowski space, whose fixed points form the real Minkowski space. Taking the real part of the projection, we have also the possibility of taking the real part of a point in the complexified Minkowski space.

For each $Z \in T$, the anti-linear functional

$$S \ni \omega \mapsto (Z|\omega) \in \mathbb{C}$$

depends only on the projection π of Z in T/S (by the isotropy of S). We have, therefore, a pairing between T/S and S given by

$$(\pi|\omega) := (Z|\omega) \qquad \text{for } \pi \in T/S,\ \omega \in S$$

where Z is any element of T whose projection on T/S is π. The pairing defines an isomorphism between T/S and $\overline{S}^*$.

It is easy to describe the affine structure of M. Any two two-dimensional subspaces $z_1, z_2 \subset T$ transversal to S define a linear map v from T/S to S,

$$\pi \mapsto v(\pi) := \frac{1}{\mathrm{i}}(Z_2(\pi) - Z_1(\pi)) \tag{1}$$

where $Z_j(\pi) \in z_j$ are twistors whose projections on T/S are equal π. In the case of isotropic subspaces this map is Hermitian:

$$(\pi|v\eta) = -\frac{1}{\mathrm{i}}(Z_1(\pi)|Z_2(\eta) - Z_1(\eta)) = -\frac{1}{\mathrm{i}}(Z_1(\pi)|Z_2(\eta))$$
$$= \frac{1}{\mathrm{i}}(Z_2(\pi) - Z_1(\pi)|Z_2(\eta)) = \overline{(\eta|v\pi)}$$

for $\pi, \eta \in T/S \cong \overline{S}^*$. Therefore any two points of M define an element of the vector space

$$S \otimes \overline{S}$$

which is Hermitian with respect to the natural conjugation $\omega \otimes \overline{\lambda} \mapsto \lambda \otimes \overline{\omega}$. We recognize here the well known spinor presentation of the *Minkowski vector space* as the real part

$$\boldsymbol{M} := \mathrm{Re}(S \otimes \overline{S}).$$

It is easy to see that Minkowski vectors $v \in \boldsymbol{M}$ can be added to points $x \in M$, by the converse procedure: to each point Z of x we add $\mathrm{i}v(\pi(Z))$. In other words, the action of translations on M is induced by the action of translations on T, defined by

$$T \ni Z \mapsto Z + \mathrm{i}v(\pi(Z)) \in T. \tag{2}$$

These are precisely transformations from G which act as identity on S. The subgroup of G composed of transformations preserving S and having determinant 1 on S, is precisely the (covering of the) Poincaré group. It acts on $\boldsymbol{M}$ via the known spinor action of the Lorentz group $SL(S) \cong SL(2, \mathbb{C})$ on S and (the complex conjugate action on) $\overline{S}$.

The whole group G acts naturally on the compactified Minkowski space. These transformations are known as conformal transformations. That is why $G \cong SU(2, 2)$ is said to be the *conformal group*.

Since we want to have a fixed metric tensor on $\boldsymbol{M}$, we assume that a fixed volume form ϵ on S is given. In terms of ϵ, the Minkowski metric is given by

$$g_{ab} = g_{AA'BB'} := \epsilon_{AB}\overline{\epsilon}_{A'B'}$$

i.e. the scalar product of two vectors $v, w \in \boldsymbol{M}$ is given by

$$g(v, w) = g_{ab}v^a w^b = \epsilon_{AB}\overline{\epsilon}_{A'B'}v^{AA'}w^{BB'}. \tag{3}$$

Small latin letters are reserved for the Minkowski vector indices, whereas the capital latin letters are used as spinor indices (and primed ones refer to the complex conjugate spinors). Recall that one vector index a corresponds to the pair of spinor indices A, A'. It is convenient to use indices to indicate the character of a tensor and the kind of operation being performed (e.g. contraction) rather than to refer to a particular basis (*abstract indices* of [5]).

From now on by a twistor space we mean the whole structure described by T, $(\cdot|\cdot)$, S and ϵ. Elements of T are called *twistors*. It is sometimes convenient to choose a fixed element $x_0 \in M$ (the *origin*), i.e. a two-dimensional isotropic subspace S_0 transversal to S. In this case

$$T = S \oplus S_0 \cong S \oplus \overline{S}^* \tag{4}$$

and we can represent a twistor $Z \in T$ by a pair of spinors (ω, π), where $\omega \in S$ and $\pi \in \overline{S}^*$. For $Z_1 = (\omega, \pi)$, $Z_2 = (\lambda, \eta)$ we have

$$(Z_1|Z_2) = \langle \omega, \overline{\eta} \rangle + \langle \pi, \overline{\lambda} \rangle = \omega^A \overline{\eta}_A + \pi_{A'}\overline{\lambda}^{A'}.$$

(Of course, if we change the reference point $x_0 \mapsto \tilde{x}_0$, then $\omega \mapsto \tilde{\omega} = \omega - \mathrm{i}v(\pi)$, where $v = \tilde{x}_0 - x_0$; in coordinates, $\tilde{\omega}^A = \omega^A - \mathrm{i}v^{AA'}\pi_{A'}$.)

The twistor space provides a natural framework for the conformal geometry of the Minkowski space. Moreover, it has a natural structure of a *phase space* of a massless particle of an arbitrary helicity (spin). Consider the following G-invariant constant symplectic 2-form on T:

$$\Omega = -2\,\mathrm{Im}(\cdot|\cdot) \qquad \text{i.e. } \Omega(Z_1, Z_2) = \mathrm{i}((Z_1|Z_2) - (Z_2|Z_1)) \qquad \text{for } Z_1, Z_2 \in T. \tag{5}$$

Denote by $r\frac{\partial}{\partial r}$ the radial vector field on T (the identity map). Since $\pounds_{r\frac{\partial}{\partial r}}\Omega = 2\Omega$,

$$\gamma := \frac{1}{2} r\frac{\partial}{\partial r} \lrcorner \Omega \tag{6}$$

is the potential of Ω, i.e. $\Omega = \mathrm{d}\gamma$. By the construction, γ is also G-invariant. Therefore, we can calculate the moment map for the action of G on (T, Ω):

$$J_X(Z) = X_T(Z) \lrcorner \gamma_Z = \tfrac{1}{2}\Omega(Z, X_T(Z)). \tag{7}$$

Here X_T is the fundamental vector field on T corresponding to an element X of the Lie algebra of G. In particular, taking as X the infinitesimal translation, $Z \mapsto X_T(Z) := \mathrm{i}v(\pi(Z))$ (see (2)), we obtain (setting $\pi(Z) \equiv \pi$)

$$J_X(Z) = \tfrac{1}{2}\Omega(Z, \mathrm{i}v(\pi)) = -\,\mathrm{Im}(Z|\mathrm{i}v(\pi)) = (\pi|v(\pi)) = (v(\pi)|\pi) = \langle v, \overline{\pi} \otimes \pi\rangle$$

hence the linear momentum equals

$$p = \overline{\pi} \otimes \pi \qquad (\text{i.e. } p_a = p_{AA'} = \overline{\pi}_A \pi_{A'}).$$

Of course, $p^2 = p_a p^a = 0$.

Poincaré transformations preserving the origin $x_0 \in M$ (i.e. the subspace S_0) are of the form

$$(\omega, \pi) \mapsto (\Lambda\omega, (\overline{\Lambda}^*)^{-1}\pi)$$

where $\Lambda \in SL(S) \cong SL(2, \mathbb{C})$. Taking as X in (7) an element of $sl(S) \cong sl(2, \mathbb{C})$, we obtain

$$\begin{aligned} J_X(Z) &= \tfrac{1}{2}\Omega(Z, (X\omega, -\overline{X}^*\pi)) = \mathrm{Im}(Z|(X\omega, -\overline{X}^*\pi)) = 2\,\mathrm{Im}\langle X\omega, \overline{\pi}\rangle \\ &= 2\,\mathrm{Im}\, X^B{}_C \omega^C \overline{\pi}_B = 2\,\mathrm{Im}\, X^{BA}\epsilon_{AC}\omega^C\overline{\pi}_B \\ &= -2\,\mathrm{Im}\, X^{BA}\omega_A\overline{\pi}_B = 2\,\mathrm{Re}\,\mathrm{i}X^{AB}\omega_A\overline{\pi}_B \end{aligned} \tag{8}$$

(we have used the symmetry of X^{AB}). Since the convenient invariant scalar product on

$$\bigwedge^2 \boldsymbol{M} \cong o(g) \cong sl(S) \cong (S \otimes S)_{\mathrm{symm}} \tag{9}$$

(here $o(g) \subset \mathrm{End}(\boldsymbol{M})$ is composed of infinitesimal Lorentz transformations) is given by

$$\langle\!\langle X, Y\rangle\!\rangle := -\tfrac{1}{2}\,\mathrm{tr}\, XY = \tfrac{1}{2}X^{jk}Y_{jk} = 2\,\mathrm{Re}\, X^{AB}Y_{AB} \tag{10}$$

we see that the Lorentz momentum (as valued in $(S \otimes S)_{\mathrm{symm}}$) is given by

$$M^{AB} = \mathrm{i}\omega^{(A}\overline{\pi}^{B)} = \frac{\mathrm{i}}{2}(\omega^A\overline{\pi}^B + \overline{\pi}^A\omega^B) \tag{11}$$

or, as a bi-vector,

$$M^{ab} := M^{AB}\overline{\epsilon}^{B'A'} + \epsilon^{BA}\overline{M}^{A'B'} = -\mathrm{i}(\omega^{(A}\overline{\pi}^{B)}\overline{\epsilon}^{A'B'} - \epsilon^{AB}\overline{\omega}^{(A'}\pi^{B')}) \tag{12}$$

(cf formula (2.7) in [2] or (1.34) in [5]). Taking $\mathrm{i}M^{AB} = -\omega^{(A}\overline{\pi}^{B)}$ instead of M^{AB}, one can easily calculate the Pauli–Lubanski *polarization* vector $W = (\mathrm{i}M)p$:

$$W^a = (\mathrm{i}M)^{ab}p_b = (\omega^{(A}\overline{\pi}^{B)}\overline{\epsilon}^{A'B'} + \epsilon^{AB}\overline{\omega}^{(A'}\pi^{B')})\overline{\pi}_B\pi_{B'} = \tfrac{1}{2}(Z|Z)p^a$$

hence

$$s := \tfrac{1}{2}(Z|Z) \tag{13}$$

is the *helicity* of the massless particle.

Warning: multiplication by i in $\overset{2}{\bigwedge} \boldsymbol{M}$ is always understood as the multiplication by i in any of three isomorphic complex spaces in (9). (It coincides with the Hodge star on $\overset{2}{\bigwedge} \boldsymbol{M}$.)

2. Natural position in the two-twistor phase space

Definition 2.1. The subset

$$\mathbf{Tp(2)} := \{(Z_1, Z_2) \in T \times T : \pi(Z_1) \text{ and } \pi(Z_2) \text{ are linearly independent}\}$$

is said to be the *two-twistor phase space*.

It is clear that two linearly independent twistors define a two-dimensional subspace of T, and the subspace is transversal to S when the two twistors have linearly independent projections on T/S. It follows that each pair $(Z_1, Z_2) \in \mathbf{Tp(2)}$ defines a point in the complexified Minkowski space. Using the identification (4), $Z_1 = (\omega, \pi)$, $Z_2 = (\lambda, \eta)$, it is easy to see that this point $z \in S \otimes \overline{S}$ is given by (cf [5])

$$z^a = z^{AA'} = \frac{\mathrm{i}}{f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}) \tag{14}$$

where $f := \pi^{A'}\eta_{A'}$ ($\mathbf{Tp(2)}$ is exactly the subset of $T \times T$ on which $f \neq 0$).

Note that the square of the total mass is strictly positive on $\mathbf{Tp(2)}$:

$$m^2 = p_a p^a = 2|f|^2 > 0 \tag{15}$$

where

$$p_a = p_{AA'} = \overline{\pi}_A \pi_{A'} + \overline{\eta}_A \eta_{A'} \tag{16}$$

is the *total linear momentum*.

Of course, the map $\mathbf{Tp(2)} \ni (Z, W) \mapsto z \in S \otimes \overline{S}$ is Poincaré covariant. In particular,

$$\{p_a, z^b\} = \delta_a{}^b. \tag{17}$$

For the real and imaginary parts of $z = X + \mathrm{i}Y$, we obtain

$$\{p_a, X^b\} = \delta_a{}^b \qquad \{p_a, Y^b\} = 0 \tag{18}$$

(hence Y is a *vector*: it does not transform under translations). A natural question about the real position X^a is whether it belongs to the world line of the massive particle whose linear momentum is p and the Lorentz momentum (with respect to the *origin*) is

$$M = M_1 + M_2 \tag{19}$$

(the total Lorentz momentum), where

$$M_1^{AB} = \mathrm{i}\omega^{(A}\overline{\pi}^{B)} \qquad M_2^{AB} = \mathrm{i}\lambda^{(A}\overline{\eta}^{B)}.$$

Recall that such a line is given by the locus of points X in the Minkowski space such that the Lorentz momentum with respect to X has only the rotational part, i.e.

$$M_X = R \tag{20}$$

where

$$M_X^{ab} = M^{ab} - (p^a X^b - X^a p^b)$$
$$R^{ab} = M^{ab} - \frac{1}{m^2}((Mp)^a p^b - p^a (Mp)^b)$$

(R does not depend on the point relative to which the Lorentz momentum is defined). It is easy to see that (20) is equivalent to $M_X p = 0$. Both conditions are equivalent to the following condition: $m^2 X + Mp$ is parallel to p. In our case (X given as the real part of z), the latter condition is satisfied because of the following formula

$$m^2 X^a = -M^{ab} p_b + l p^a \qquad l := \mathrm{Im}(\omega^A \overline{\pi}_A + \lambda^A \overline{\eta}_A) \tag{21}$$

shown in the appendix (section A.3; cf also [2, 5]), hence X indeed belongs to the *centre-of-mass* world line. In the appendix we show also that

$$m^2 Y^a = W^a - (s_1 + s_2) p^a \tag{22}$$

where $W := (\mathrm{i}M)p$ is the total Pauli–Lubanski vector and $s_j = \frac{1}{2}(Z_j|Z_j)$ are the helicities of our two twistors.

Lemma 2.2.

$$\{X^a, X^b\} = \{Y^a, Y^b\} = \frac{1}{m^4}\{W^a, W^b\} = -\frac{1}{m^2} R^{ab}. \tag{23}$$

Proof. The first equality follows from $\{z^a, z^b\} = 0$ (z is a holomorphic function of (Z, W) and the symplectic structure is of type $(1, 1)$). The second equality is the consequence of (22) and $\{W^a, p^b\} = 0$ (W is a vector: it does not change under translations). The third equality follows from

$$\begin{aligned}\{(\mathrm{i}M)^{jl}, (\mathrm{i}M)^{km}\} p_l p_m = -\{M^{jl}, M^{km}\} p_l p_m &= -p^2 M^{jk} + (Mp)^j p^k - (Mp)^k p^j \\ &= -\, p^2 R^{jk}\end{aligned} \tag{24}$$

and

$$p_l \{(\mathrm{i}M)^{jl}, p_m\} \sim p_l \epsilon^{jlab}\{M_{ab}, p_m\} = p_l \epsilon^{jlab}(g_{am} p_b - g_{bm} p_a) = 0 \tag{25}$$

(ϵ^{jlab} is the totally antisymmetric tensor). □

3. Commuting positions

Apart from the natural position X in **Tp(2)**, one can introduce two other natural positions—associated with the two massless constituents.

Although a massless particle with spin $s \neq 0$ has no world line (points x in the Minkowski space such that $M_x p = 0$ form a three-dimensional hyperplane; here (p, M) is the Poincaré momentum of the particle), it determines a world line with respect to any fixed four-velocity u_0 (an observer):

$$M_x u_0 = 0 \qquad \text{or} \qquad x = \frac{1}{p \cdot u_0}(-M u_0 + (x \cdot u_0) p) \tag{26}$$

(we use a short notation $x \cdot y$ for the scalar product).

In our case of two massless particles with Poincaré momenta (p_j, M_j), $k = 1, 2$, we have at our disposal the total linear momentum $p = p_1 + p_2$, which we can use as the reference. Thus, we are led to the following 'world lines relative to the common rest frame':

$$x_j = \frac{1}{p_j \cdot p}(-M_j p + (x_j \cdot p) p_j) \qquad j = 1, 2. \tag{27}$$

Moreover, on each of these lines one can choose a point by requiring that $(x_j - X) \cdot p = 0$ (i.e. x_j and X have the same time relative to p). We obtain then the final formula for the two new positions:

$$x_j = \frac{1}{p_j \cdot p}(-M_j p + (X \cdot p) p_j) \qquad j = 1, 2. \tag{28}$$

Note that

$$X = \frac{x_1 + x_2}{2} \tag{29}$$

and $\Delta x_j := x_j - X$ satisfy $\Delta x_1 = -\Delta x_2$, $(\Delta x_1) \cdot p = 0$ and

$$\Delta x_1 = \frac{1}{m^2}((M_2 - M_1)p + l(p_1 - p_2)) \tag{30}$$

($l = X \cdot p$ is given in (21)).

We now derive a simple (twistor) formula for Δx_1. As before, $Z_1 = (\omega, \pi)$ and $Z_2 = (\lambda, \eta)$.

Lemma 3.1. We have

$$m^2 \Delta x_1 = 2 \operatorname{Im} \overline{\rho} w \tag{31}$$

where $\rho = (Z_1 | Z_2)$, $w^a = \overline{\eta}^A \pi^{A'}$.

Proof. First we shall show that $(\Delta x_1) \cdot p_1 = 0$. From (30) we see that this holds if and only if

$$p_1 \cdot (M_2 - M_1) p = l p_1 \cdot p_2. \tag{32}$$

Since

$$p_2 \cdot M_1 p_1 = -\mathrm{i} \overline{\eta}_A \eta_{A'} (\omega^{(A} \overline{\pi}^{B)} \overline{\epsilon}^{A'B'} - \epsilon^{AB} \overline{\omega}^{(A'} \pi^{B')}) \overline{\pi}_B \pi_{B'} = (p_1 \cdot p_2) \operatorname{Im}(\omega^B \overline{\pi}_B)$$

and, analogously, $p_1 \cdot M_2 p_2 = (p_1 \cdot p_2) \operatorname{Im}(\lambda^B \overline{\eta}_B)$, we have

$$p_1 \cdot (M_2 - M_1) p_2 = p_1 M_2 p_2 + p_2 M_1 p_1 = (p_1 \cdot p_2) \operatorname{Im}(\omega^B \overline{\pi}_B + \lambda^B \overline{\eta}_B)$$

hence (32) definitely holds.

It follows that Δx_1 is perpendicular both to p_1 and p_2. It is easy to see that the complex vector w has also this property. Therefore there exists a complex number α such that $\Delta x_1 = \overline{\alpha} w + \alpha \overline{w}$. One can easily compute

$$w \cdot M_1 p = \mathrm{i} |f|^2 \omega^A \overline{\eta}_A \qquad w \cdot M_2 p = -\mathrm{i} |f|^2 \pi_{A'} \overline{\lambda}^{A'} \qquad w \cdot w = 0 \qquad w \cdot \overline{w} = -|f|^2$$

hence

$$-\frac{\mathrm{i}}{2} \rho = \frac{1}{m^2} w \cdot (M_2 - M_1) p = w \cdot (\Delta x_1) = \alpha w \cdot \overline{w} = -\alpha |f|^2.$$

It follows that $\alpha = \mathrm{i} \rho / m^2$ and this ends the proof. □

The main result of this paper is formulated in the following theorem.

Theorem 3.2. Both positions introduced in formula (28) have commuting coordinates:

$$\{x_1^a, x_1^b\} = 0 \qquad \{x_2^a, x_2^b\} = 0.$$

Proof. We introduce the following tensor notation

$$(\{t \overset{\otimes}{,} s\})^{ab} := \{t^a, s^b\}$$

for the bracket of two quantities with indices.

In the appendix we prove that

$$\{X \overset{\otimes}{,} \Delta x_1\} + \{\Delta x_1 \overset{\otimes}{,} X\} = 0 \tag{33}$$

(section A.5) and

$$\{\Delta x_1 \overset{\otimes}{,} \Delta x_1\} = -\{X \overset{\otimes}{,} X\} \tag{34}$$

(section A.6), hence

$$\{(X \pm \Delta x_1) \overset{\otimes}{,} (X \pm \Delta x_1)\} = [\{X \overset{\otimes}{,} X\} + \{\Delta x_1 \overset{\otimes}{,} \Delta x_1\}] \pm [\{X \overset{\otimes}{,} \Delta x_1\} + \{\Delta x_1 \overset{\otimes}{,} X\}] = 0.$$

□

4. The structure of the two-twistor phase space

Due to the commutativity of the coordinates of $x := x_1$, we can treat (x^a, p_b) as the usual canonical variables:

$$\{x^a, x^b\} = 0 \qquad \{p^a, p^b\} = 0 \qquad \{p^a, x_b\} = \delta^a_b$$

(the last bracket follows from the fact that x is Poincaré covariant).

From [1] we know, that $\Sigma := M_x$ commutes both with x and p. In the appendix (section A.7) we prove the following lemma.

Lemma 4.1. As an element of $(S \otimes S)_{\mathrm{symm}}$, Σ is given by

$$\Sigma = \overline{\eta} \otimes \overline{\sigma} + \overline{\sigma} \otimes \overline{\eta} \qquad \text{where } \sigma := \frac{\mathrm{i}}{2f}((s_1 - s_2)\pi + \rho\eta). \tag{35}$$

Note that Σ determines η up to a complex factor. It follows that the complex projective part of η, hence also the real projective part of p_2, commutes with x (of course, the whole η commutes with p). But we have in fact more than this: in fact the *real projective part* $[\eta] \in \mathbb{RP}(T/S)$ of η commutes with x, due to the following formula

$$\{\eta^A, x^b\} = \frac{1}{m^2}\eta^A p_1^b \tag{36}$$

which we prove in the appendix (section A.8). By the *real projective* space of a complex vector space E we shall mean the quotient of $E \setminus \{0\}$ under the equivalence relation

$$\zeta' \sim \zeta \iff \zeta' = t\zeta \qquad \text{for some positive real number } t. \tag{37}$$

Let us now distinguish the following three groups of variables:

(1) (x, p), describing T^*_+M—the *forward* cotangent bundle to the Minkowski space (by this we mean the subset of T^*M such that $p^2 > 0$).

(2) (ϕ, e), where

$$\phi := \text{phase of } f \text{ (defined modulo } 2\pi) \qquad e := 2s_1 \tag{38}$$

Note that

$$\{e, \phi\} = 1. \tag{39}$$

(3) Σ together with the real-projective part $[\eta]$ of η.

Proposition 4.2. The above three groups of variables are mutually commuting.

Proof. We know already that the third group commutes with the first. It is sufficient to show that s_1 and ϕ commute with the first and the third group.

Since variables x, p, η, Σ are all invariant with respect to the change of the phase of the first twistor, they commute with s_1 (for Σ, we can use either the invariance of σ, or the definition $\Sigma = M_x = M - p \wedge x$ and the fact that M commutes with each conformal invariant).

Trivially, ϕ commutes with p and η (all are 'π-like'). It commutes with Σ, since f is Lorentz-invariant (and Σ generates the Lorentz group action on 'π-like' variables). It also commutes with x. Namely, using easy to check formulae

$$\{f, l\} = 0 \qquad \{f, \rho\} = 0 \qquad \{f, \overline{\rho}\} = 0 \tag{40}$$

we have

$$\{f, X\} = \frac{1}{m^2}\{f, l\}p = 0 \qquad \text{and} \qquad \{f, \Delta x_1\} = \frac{1}{\mathrm{i}m^2}(\{f, \overline{\rho}\}w - \{f, \rho\}\overline{w}) = 0. \tag{41}$$

□

Theorem 4.3. The two-twistor phase space is isomorphic to the symplectic product

$$T^*_+ M \times T^*S^1 \times T^*N \tag{42}$$

where $N := \mathbb{RP}(S)$ (S is the spinor space).

Proof. N is of course the quotient of $S \setminus \{0\}$ by the equivalence relation (37). We denote the equivalence class of $\zeta \in S$ by $[\zeta]$.

Let the *real* duality between S^* and S be defined by

$$\langle \xi, \zeta \rangle := 2\,\mathrm{Re}\langle \xi, \zeta \rangle_{\mathbb{C}} \qquad \text{where } \langle \xi, \zeta \rangle_{\mathbb{C}} := \xi_A \zeta^A. \tag{43}$$

Since T^*N is the special symplectic reduction of $T^*(S \setminus \{0\}) \subset T^*S$ arising from the natural projection, the elements of T^*N are in one-to-one correspondence with classes of elements $(\zeta, \xi) \in T^*(S \setminus \{0\}) = (S \setminus \{0\}) \times S^*$ modulo the equivalence

$$(\zeta, \xi) \sim \left(t\zeta, \frac{1}{t}\xi\right) \qquad t > 0 \tag{44}$$

with additional condition

$$\langle \xi, \zeta \rangle = 0. \tag{45}$$

The class of (ζ, ξ) is a covector at the point $[\zeta] \in \mathbb{RP}(S)$.

To each $X \in sl(S)$ there corresponds the vector field $\zeta \mapsto X\zeta$ on S and its canonical Hamiltonian on T^*S

$$J_X(\zeta, \xi) = \langle \xi, X\zeta \rangle = 2\,\mathrm{Re}(\xi_A X^A{}_B \zeta^B) = -2\,\mathrm{Re}(\xi_A X^{AB} \zeta_B). \tag{46}$$

It follows that the corresponding moment map (as valued in $(S \otimes S)_{\mathrm{symm}}$) is given by

$$(\zeta, \xi) \mapsto -\tfrac{1}{2}(\zeta \otimes \xi + \xi \otimes \zeta). \tag{47}$$

(ξ on the right-hand side is considered as an element of S.) Since the action of the Lorentz group on $S \setminus \{0\}$ passes to the quotient N, the same formula is valid on T^*N (of course (47) is invariant with respect to the scaling (44)).

We may now identify the target space of the third group of variables. We can pass from $[\eta]$ to $[\zeta] := [\overline{\eta}] \in \mathbb{RP}(S)$ and use the identification of the Lorentz moment maps (35) and (47) to identify

$$\xi = -2\overline{\sigma}. \tag{48}$$

Our mutually commuting three groups of variables clearly describe a map

$$(\omega, \pi, \lambda, \eta) \mapsto (x, p; \phi, e; \zeta, \xi) \tag{49}$$

from **Tp(2)** to (42). We shall show that this map is surjective (assuming, of course, that $\zeta \neq 0$, $\langle \xi, \zeta \rangle = 0$ and (ζ, ξ) is considered up to equivalence (44)). Suppose that we are given $(x, p; \phi, e; \zeta, \xi)$. Define $\eta := t\overline{\zeta}$, where $t > 0$ is such that

$$\overline{\eta} \otimes \eta = p_2 \qquad \text{satisfies } p \cdot p_2 = \frac{m^2}{2}$$

(we shall assume in the sequel that ζ is adjusted so that $t = 1$). Define π such that

$$\overline{\pi} \otimes \pi = p - p_2 \qquad \text{and} \qquad \pi^{A'} \eta_{A'} = f := \mathrm{e}^{\mathrm{i}\phi} \sqrt{\frac{m^2}{2}}.$$

Due to (35) and (48), we have to set

$$\rho := \overline{\pi}^A \xi_{A'} \qquad s_2 = e/2 + (s_2 - s_1) = e/2 + \mathrm{i}\overline{\eta}^A \xi_A. \tag{50}$$

Thus

$$\Delta x_1 := \frac{1}{\mathrm{i}m^2} (\overline{\rho} w - \rho \overline{w}) \qquad \text{where } w = \overline{\eta} \otimes \pi \tag{51}$$

and

$$X := x - \Delta x_1 \qquad Y := \frac{2}{m^2} (\mathrm{Re}\, \overline{\rho} w - (e/2 + \mathrm{i}\overline{\eta}^A \xi_A) p_1 - e/2 p_2) \tag{52}$$

and finally

$$z := X + \mathrm{i}Y \qquad \omega^A := \mathrm{i}z^{AA'} \pi_{A'} \qquad \lambda^A := \mathrm{i}z^{AA'} \eta_{A'}. \tag{53}$$

One can check that these definitions are consistent. For example, s_1 and s_2 are indeed given by formula (13) applied to the first and the second twistor, respectively. Moreover, each definition was forced by the considered map, which shows that the map is also injective. □

5. Final remarks

By theorem 4.3, the two-twistor phase space is a product of the 'charge phase space' T^*S^1 and a type of extended phase space as considered in [1]. Spaces considered in [1] involve one Lorentz co-adjoint orbit, while the present model is a 'direct integral' of cases with different orbits. Using (35) and (59) we can calculate the parameter of the Lorentz co-adjoint orbits (cf [1]) which occur here:

$$(a + \mathrm{i}b)^2 = \langle\langle \Sigma, \Sigma \rangle\rangle_{\mathbb{C}} = -4(\epsilon(\overline{\eta}, \overline{\sigma}))^2 = (s_1 - s_2)^2.$$

It follows that $b = 0$. (Note that the Killing forms (10) and (58) differ by the sign from those considered in [1]; this is also the reason for the change of sign at R in (23).)

Another way of writing (42) would be

$$\mathbf{Tp(2)} \cong T^*_+(M \times S^1 \times \mathbb{RP}(S)) \tag{54}$$

('+' refers to $p^2 > 0$), hence the two twistor phase space is just (a subset of) a cotangent bundle, which allows to consider not only Hamiltonian but also Lagrangian formulation of dynamics (in this connection, see e.g. [7, 8]).

Next possibility would be to consider

$$T^*(M \times S^1 \times S) \tag{55}$$

which involves all Lorentz orbits. It seems that such a space should emerge as a symplectic factor in the three- (or more) twistor space.

Further study is required to explain whether the new insight into the structure of the two-twistor phase space can be useful for a Hamiltonian description of a charged particle with magnetic moment.

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Appendix

A.1. Useful spinorial formulae

(1) metric: $g_{ab} = \epsilon_{AB}\overline{\epsilon}_{A'B'}$
(2) inverse ϵ: $\epsilon^{AB}\epsilon_{AC} = \delta^{B}{}_{C}$
(3) raising–lowering: $\xi^A = \epsilon^{AB}\xi_B$, $\xi_B = \xi^A\epsilon_{AB}$
(4) anti-symmetrization:

$$2\omega^{[A}\lambda^{B]} \equiv \omega^A\lambda^B - \lambda^A\omega^B = \epsilon(\omega,\lambda)\epsilon^{AB} \equiv (\omega_C\lambda^C)\epsilon^{AB} \tag{56}$$

(5) isomorphism (9): Relations between M^{ab}, $M^a{}_b$, $M^A{}_B$ and M^{AB} are as follows:

$$\begin{aligned} &M^a{}_b = M^{ac}g_{cb} \qquad M^A{}_B = M^{AC}\epsilon_{CB} \\ &M^{ab} := M^{AB}\overline{\epsilon}^{B'A'} + \epsilon^{BA}\overline{M}^{A'B'} \qquad 2M^{AB} = M^{AA'B}{}_{A'}. \end{aligned} \tag{57}$$

In addition to formula (10), we have the complex Killing form

$$\langle\!\langle X, Y\rangle\!\rangle_{\mathbb{C}} := \langle\!\langle X, Y\rangle\!\rangle - \mathrm{i}\langle\!\langle \mathrm{i}X, Y\rangle\!\rangle = 2X^{AB}Y_{AB} \tag{58}$$

(cf [1]). For $X^{AB} = u^{(A}v^{B)}$ we have

$$\langle\!\langle X, \rangle\!\rangle_{\mathbb{C}} = -(\epsilon(u,v))^2. \tag{59}$$

A.2. Twistor Poisson brackets

The symplectic form (5) has the following coordinate description

$$\Omega = \mathrm{i}(d\omega^A \wedge d\overline{\pi}_A + d\pi_{A'} \wedge d\overline{\omega}^{A'})$$

hence we have

$$\{\overline{\pi}_A, \omega^B\} = \mathrm{i}\delta^B_A \qquad \{\overline{\omega}^{A'}, \pi_{B'}\} = \mathrm{i}\delta^{A'}_{B'}$$

and the remaining two combinations are zero. It is also convenient to record

$$\{\overline{\pi}_A, \omega_B\} = \{\omega_A, \overline{\pi}_B\} = \mathrm{i}\epsilon_{AB}.$$

When computing various Poisson brackets in the two-twistor space, it is convenient to remember the following simplifying rules.

(1) Conformal invariants are also Poincaré invariants hence commute with any function of Poincaré generators, e.g. $\{\rho, m^2\} = 0$.

(2) Purely holomorphic (resp. antiholomorphic) functions commute among themselves. Due to this, we have for instance $\{\rho, w\} = 0$.

(3) Purely momentum functions commute, e.g. $\{m^2, w\} = 0$, $\{w \overset{\otimes}{,} w\} = 0$, $\{w \overset{\otimes}{,} \overline{w}\} = 0$.

(4) Helicity generates rotations by phase factors, hence it commutes with any quantity invariant under changing the phase of the twistor, e.g. $\{s_1, \overline{\rho} w\} = 0$.

A.3. Proof of formulae (21) and (22)

The N-twistor analogue of (14) is given by

$$m^2 z^a = 2\mathrm{i}(\overline{\pi}^{jB}\overline{\pi}^k_B)\omega^A_j \pi^{A'}_k$$

where the indices $j, k = 1, \ldots, N$ label the $N \geqslant 2$ twistors. We have

$$\begin{aligned}
m^2 z^a &= 2\mathrm{i}(\omega^{(A}_j \overline{\pi}^{jB)} + \omega^{[A}_j \overline{\pi}^{jB]})\overline{\pi}^k_B \pi^{A'}_k = 2M^{AB}_j \overline{\pi}^k_B \pi^{A'}_k - \mathrm{i}(\omega^C_j \overline{\pi}^j_C)\epsilon^{AB}\overline{\pi}^k_B \pi^{A'}_k \\
&= (M^{AB}_j \overline{\pi}^k_B \pi^{A'}_k + \overline{M}^{A'B'}_j \pi^k_{B'} \overline{\pi}^A_k) + \frac{1}{\mathrm{i}}(\mathrm{i}M^{AB}_j \overline{\pi}^k_B \pi^{A'}_k \\
&\quad + \overline{\mathrm{i}M}^{A'B'}_j \pi^k_{B'} \overline{\pi}^A_k) - \mathrm{i}(\omega^C_j \pi^j_C) p^{AA'} \\
&= -M^{ab} p_b + \mathrm{i}(\mathrm{i}M)^{ab} p_b - \mathrm{i}(\omega^C_j \pi^j_C) p^a.
\end{aligned}$$

Taking the real and imaginary part of it, we get

$$m^2 X^a = -M^{ab} p_b + \mathrm{Im}(\omega^C_j \pi^j_C) p^a \qquad m^2 Y^a = (\mathrm{i}M)^{ab} p_b - \mathrm{Re}(\omega^C_j \pi^j_C) p^a. \tag{60}$$

Note that $(\mathrm{i}M)p =: W$ is the Pauli–Lubanski vector and $\mathrm{Re}(\omega^C_j \pi^j_C) = \sum_j s_j$ is the sum of helicities. We have therefore

$$Y = \frac{1}{m^2}\left(W - \left(\sum_j s_j\right)p\right). \tag{61}$$

A.4. Pauli–Lubanski vector

Here we collect certain useful algebraic formulae which allow for yet another expression for Y and W.

A.4.1. Some scalar products with w.

$$w \cdot w = 0 \qquad w \cdot \overline{w} = -|f|^2 \qquad p_1 \cdot w = 0 \qquad p_2 \cdot w = 0$$

$$Y \cdot w = -\frac{\rho}{2} \qquad Y \cdot \overline{\rho} w = -\tfrac{1}{2}|\rho|^2 \qquad Y \cdot \mathrm{Re}\,\overline{\rho} w = -\tfrac{1}{2}|\rho|^2 \qquad Y \cdot \mathrm{Im}\,\overline{\rho} w = 0 \tag{62}$$

$$\overline{\rho} w \cdot \overline{\rho} w = 0 \qquad \overline{\rho} w \cdot \overline{\overline{\rho} w} = -|\rho|^2 |f|^2 \qquad \mathrm{Re}\,\overline{\rho} w \cdot \mathrm{Im}\,\overline{\rho} w = 0 \tag{63}$$

$$(\mathrm{Re}\,\overline{\rho} w)^2 = (\mathrm{Im}\,\overline{\rho} w)^2 = \tfrac{1}{2}(\overline{\rho} w \cdot \overline{\overline{\rho} w}) = -\frac{|\rho|^2 |f|^2}{2}.$$

The last formula in (62) states that $Y \cdot \Delta x_1 = 0$. Hence Y must be a linear combination of p_1, p_2 and $\mathrm{Re}\,\overline{\rho} w$ (the latter vector is perpendicular to Δx_1 by (63)):

$$Y = c_1 p_1 + c_2 p_2 + c_3\, \mathrm{Re}\,\overline{\rho} w.$$

We have

$$c_2 p_1 \cdot p_2 = p_1 \cdot Y = \operatorname{Im} \overline{\pi}^A \pi^{A'} \frac{\mathrm{i}}{f} (\omega^A \eta^{A'} - \lambda^A \pi^{A'}) = -\operatorname{Re} \omega^A \overline{\pi}_A = -s_1$$

$$c_1 p_1 \cdot p_2 = p_2 \cdot Y = (p - p_1) \cdot Y = -(s_1 + s_2) - (-s_1) = -s_2$$

and

$$-c_3 \frac{|\rho|^2 |f|^2}{2} = c_3 (\operatorname{Re} \rho w)^2 = \operatorname{Re} \overline{\rho} w \cdot Y = -\tfrac{1}{2} |\rho|^2$$

hence

$$|f|^2 c_1 = -s_2 \qquad |f|^2 c_2 = -s_1 \qquad |f|^2 c_3 = 1$$

and finally

$$Y = \frac{1}{|f|^2} (\operatorname{Re} \overline{\rho} w - s_2 p_1 - s_1 p_2) \tag{64}$$

(cf [6]). Using (22), we see that

$$W = (s_1 + s_2) p + m^2 Y = (s_1 - s_2)(p_1 - p_2) + \overline{\rho} w + \rho \overline{w} \tag{65}$$

(cf [6]).

A.5. Proof of formula (33)

First we show that

$$\{X \overset{\otimes}{,} w\} = \frac{1}{m^2} w \wedge p. \tag{66}$$

From definitions of z and w we obtain

$$\{z^a, w^b\} = \frac{1}{f} \epsilon^{AB} \pi^{A'} \pi^{B'} \qquad \{\overline{z}^a, w^b\} = -\frac{1}{\overline{f}} \overline{\eta}^A \overline{\eta}^B \overline{\epsilon}^{A'B'}.$$

Using

$$\overline{\epsilon}^{A'B'} = -\frac{2\pi^{[A'} \eta^{B']}}{f}$$

(cf (56)), we have

$$\begin{aligned}
\{X^a, w^b\} &= \frac{1}{2f} \epsilon^{AB} \pi^{A'} \pi^{B'} - \frac{1}{2\overline{f}} \overline{\eta}^A \overline{\eta}^B \overline{\epsilon}^{A'B'} \\
&= \frac{1}{m^2} [\overline{\eta}^A \overline{\eta}^B (\pi^{A'} \eta^{B'} - \eta^{A'} \pi^{B'}) - (\overline{\pi}^A \overline{\eta}^B - \overline{\eta}^A \overline{\pi}^B) \pi^{A'} \pi^{B'}] \\
&= \frac{1}{m^2} (w^a p_2^b - p_2^a w^b - p_1^a w^b + w^a p_1^b)
\end{aligned}$$

and (66) follows. Now, using $\{X, \rho\} = 0$ (cf [2]) and

$$\begin{aligned}
\left\{X \overset{\otimes}{,} \frac{w}{m^2}\right\} &= \left\{X, \frac{1}{m^2}\right\} \otimes w + \frac{1}{m^2} \{X \overset{\otimes}{,} w\} \\
&= \frac{2}{m^4} p \otimes w + \frac{1}{m^4} w \wedge p = \frac{1}{m^4} (p \otimes w + w \otimes p)
\end{aligned}$$

we see that $\{X, \Delta x_1\}$ is symmetric. This implies (33).

A.6. Proof of formula (34)

It is easy to derive (subsequently) the following Poisson brackets

$$\{\overline{\rho}, \rho\} = 2\mathrm{i}(s_2 - s_1) \qquad \{\rho, w\} = 0 \qquad \{\overline{\rho}, w\} = \mathrm{i}(p_2 - p_1)$$

$$\{\overline{\rho}w \overset{\otimes}{,} \overline{\rho}w\} = (\mathrm{i}\overline{\rho}w) \wedge (p_2 - p_1) \qquad \{\overline{\rho}w \overset{\otimes}{,} \rho\overline{w}\} = 2\mathrm{i}(s_2 - s_1)w \otimes \overline{w}$$

$$\{\overline{\rho}w \overset{\otimes}{,} \overline{\rho}w\} + \{\rho\overline{w} \overset{\otimes}{,} \rho\overline{w}\} = \mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) \tag{67}$$

$$\{\overline{\rho}w \overset{\otimes}{,} \rho\overline{w}\} + \{\rho\overline{w} \overset{\otimes}{,} \overline{\rho}w\} = 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}. \tag{68}$$

From this we obtain

$$\{(\overline{\rho}w - \rho\overline{w}) \overset{\otimes}{,} (\overline{\rho}w - \rho\overline{w})\} = \mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) - 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w} \tag{69}$$

$$\{(\overline{\rho}w + \rho\overline{w}) \overset{\otimes}{,} (\overline{\rho}w + \rho\overline{w})\} = \mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) + 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}. \tag{70}$$

Using (69) we have

$$\{\Delta x_1 \overset{\otimes}{,} \Delta x_1\} = \left\{\frac{1}{\mathrm{i}m^2}(\overline{\rho}w - \rho\overline{w}) \overset{\otimes}{,} \frac{1}{\mathrm{i}m^2}(\overline{\rho}w - \rho\overline{w})\right\} = -\frac{1}{m^4}\{(\overline{\rho}w - \rho\overline{w}) \overset{\otimes}{,} (\overline{\rho}w - \rho\overline{w})\}$$
$$= -\frac{1}{m^4}[\mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) - 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}].$$

On the other hand, computing

$$\{p_1 - p_2, \rho\} = \{2p_1 - p, \rho\} = 2\{p_1, \rho\} = 2\mathrm{i}w$$
$$\{(p_1 - p_2) \overset{\otimes}{,} (\overline{\rho}w + \rho\overline{w})\} = 2\mathrm{i}w \wedge \overline{w}$$

and using (65) and (70), we get

$$\{W \overset{\otimes}{,} W\} = \{(\overline{\rho}w + \rho\overline{w}) \overset{\otimes}{,} (\overline{\rho}w + \rho\overline{w})\} + (s_1 - s_2)(\{(p_1 - p_2) \overset{\otimes}{,} (\overline{\rho}w + \rho\overline{w})\}$$
$$+\{(\overline{\rho}w + \rho\overline{w}) \overset{\otimes}{,} (p_1 - p_2)\})$$
$$= \mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) - 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}. \tag{71}$$

Thus,

$$\{\Delta x_1 \overset{\otimes}{,} \Delta x_1\} = -\frac{1}{m^4}\{W \overset{\otimes}{,} W\} = -\{X \overset{\otimes}{,} X\}.$$

A.7. Proof of lemma (4.1)

Since $\Sigma = M_x = M_X - p \wedge (x - X) = R + \Delta x_1 \wedge p$ and

$$m^2 R = -\{W \overset{\otimes}{,} W\} = -\mathrm{i}(\overline{\rho}w - \rho\overline{w}) \wedge (p_2 - p_1) + 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}$$

(cf (71)), we have

$$m^2\Sigma = 2\,\mathrm{Im}\,\overline{\rho}w \wedge (2p_2) + 2\mathrm{i}(s_2 - s_1)w \wedge \overline{w}$$

(recall that $m^2\Delta x_1 = 2\,\mathrm{Im}\,\overline{\rho}w$). Now,

$$[(\overline{\rho}w - \rho\overline{w}) \wedge p_2]^{AA'BB'} = (\overline{\rho}\eta^A\pi^{A'} - \rho\overline{\pi}^A\eta^{A'})\overline{\eta}^B\eta^{B'} - \overline{\eta}^A\eta^{A'}(\overline{\rho}\eta^B\pi^{B'} - \rho\overline{\pi}^B\eta^{B'})$$

hence (cf (57))

$$2[(\overline{\rho}w - \rho\overline{w}) \wedge p_2]^{AB} = (\overline{\rho}\eta^A\pi^{A'} - \rho\overline{\pi}^A\eta^{A'})\overline{\eta}^B\eta_{A'} - \overline{\eta}^A\eta^{A'}(\overline{\rho}\eta^B\pi_{A'} - \rho\overline{\pi}^B\eta_{A'})$$
$$= 2\overline{\rho}f\overline{\eta}^A\overline{\eta}^B.$$

Similarly,

$$2[w \wedge \overline{w}]^{AB} = \overline{\eta}^A\pi^{A'}\overline{\pi}^B\eta_{A'} - \overline{\pi}^A\eta^{A'}\overline{\eta}^B\pi_{A'} = f(\overline{\eta}^A\overline{\pi}^B + \overline{\pi}^A\overline{\eta}^B).$$

Finally

$$\mathrm{i}m^2\Sigma = 2f\overline{\rho\eta} \otimes \overline{\eta} + f(s_1 - s_2)(\overline{\eta} \otimes \overline{\pi} + \overline{\pi} \otimes \overline{\eta})$$
$$= f[\overline{\eta} \otimes (\overline{\rho\eta} + (s_1 - s_2)\overline{\pi}) + (\overline{\rho\eta} + (s_1 - s_2)\overline{\pi}) \otimes \overline{\eta}].$$

A.8. Proof of the formula (36)

We have

$$\{\overline{\eta}^A, X^b\} = \left\{\overline{\eta}^A, \frac{z^b + \overline{z}^b}{2}\right\} = \tfrac{1}{2}\{\overline{\eta}^A, z^b\} = \frac{\mathrm{i}}{2f}\{\overline{\eta}^A, \omega^B \eta^{B'} - \lambda^B \pi^{B'}\}$$
$$= \frac{1}{2f}\epsilon^{AB}\pi^{B'} = -\frac{1}{m^2}(\overline{\pi}^A\overline{\eta}^B - \overline{\eta}^A\overline{\pi}^B)\pi^{B'} = \frac{1}{m^2}(\overline{\eta}^A p_1^b - \overline{\pi}^A w^b)$$

and

$$\{\overline{\eta}^A, \Delta x_1^b\} = \frac{1}{\mathrm{i}m^2}\{\overline{\eta}^A, \overline{\rho} w^b - \rho \overline{w}^b\} = \frac{1}{\mathrm{i}m^2}\{\overline{\eta}^A, \overline{\rho}\} w^b = \frac{1}{m^2}\overline{\pi}^A w^b.$$

Adding these two results and taking the complex conjugation we get (36).

Remark 5.1. Similar computation gives

$$\{\pi^A, x^b\} = \frac{1}{m^2}(\pi^A p_2^b - 2\eta^A w^b).$$

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